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A VERSATILE MARKOVIAN POINT PROCESS

by

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Several qualitative features of point processes, such as certain types of fluctuations, grouping, interruptions and the inhibition of arrivals by bunch inputs can be modelled in a way which remains computationally tractable.

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1. Introduction

While the general theory of point processes on the real line has seen major developments during the recent years, the cases where analytically or algorithmically tractable results are obtained remain intimately related to an underlying Markovian assumption. Stochastic models which require as few as two general renewal processes to be considered are in most cases intractable. Standard constructions, such as the superposition of two renewal processes, destroy the regenerative properties to such an extent, that an exact, non-asymptotic analysis of the model becomes very difficult, if not impossible.

This problem has arisen early in a variety of practical situations. Notably in communications engineering, a variety of techniques have been developed by which a complicated point process is replaced by a simpler one. The latter does not share the complex probability structure of the former, but exhibits certain qualitative features of it and a few lower order moments can usually be matched by adjustment of parameters, so that the two processes are at least macroscopically very similar. A survey of such techniques may be found e.g. in L. Kosten [4].

In this paper, we shall discuss a versatile class of point processes which is closely related to finite Markov processes. Its theoretical analysis is therefore elementary and its relation to finite systems of differential equations guarantees that, at least for a wide range of parameter values, the resulting equations may readily be solved numerically.

We shall give a general description of the point process in

Section 3. It involves a very large number of parameters, but in any given application most of these can be eliminated or assigned simple values, depending on the qualitative features of the point process which the user wishes to bring forth in the model.

A crucial underlying notion is that of a probability distribution of phase type (PH-distribution), investigated in [8-12]. The next section reviews the definition and key properties of PH-distributions.

2. The PH-distributions

We shall only review the continuous PH-distributions on $[0,\infty)$, bearing in mind that there is an entirely analogous development for discrete PH-distributions, derived from the theory of finite Markov chains. All the definitions and derivations in this paper may therefore be modified to define a point process on the discrete time lattice.

Consider an (m+1)-state Markov process with m transient states and one absorbing state. Its infinitesimal generator Q is then of the form

$$Q = \begin{vmatrix} T & \underline{T}^{\circ} \\ \underline{0} & 0 \end{vmatrix}, \tag{1}$$

where T is an $m \times m$ matrix, with $T_{ii}<0$, $T_{ij} \geq 0$, for $i \neq j$ and such that T^{-1} exists. The vector $\underline{T}^{\circ}$ has nonnegative entries and satisfies $T\underline{e}+\underline{T}^{\circ}=\underline{0}$, where $\underline{e}=(1,\ldots,1)'$. A vector $(\underline{\alpha}, \alpha_{m+1})$ of initial probabilities is also given and satisfies $\underline{\alpha}\underline{e}+\alpha_{m+1}=1$, $0 \leq \alpha_{m+1}<1$.

The probability distribution $F(\cdot)$ of the time till absorption in the state m+1 is then given by

$$F(x)=1-\underline{\alpha}\exp(Tx)\underline{e}, \qquad \text{for } x\geq 0. \qquad (2)$$

The pair ($\underline{\alpha}$,T) is called <u>a representation</u> of F(·) and any probability distribution F(·) which can be so constructed is a PH-distribution. In this paper, we may assume without loss of generality that $\alpha_{m+1}=0$, so that F(·) does not have a jump at 0.

We further consider the matrices $T°$ with $T°_{ij}=T°_i$ and $A°$ with $A°=\text{diag}(\alpha_1,\ldots,\alpha_m)$. The Markov process with the infinitesimal generator

$$Q^*=T+T°A°, \qquad (3)$$

is now of considerable importance. We may always assume that C* is <u>irreducible</u>, if necessary after deletion of superfluous states from the chain Q. The matrix Q* describes the Markov chain, obtained by resetting the original chain instantaneously using the same initial probabilities, whenever an absorption into the state m+1 occurs.

The stationary probability vector $\underline{\pi}$ of Q* is obtained by solving the equations $\underline{\pi}Q^*=\underline{0}$, $\underline{\pi}\underline{e}=1$. The times of absorption (and resetting) are readily seen to form a renewal process with the underlying probability distribution F(·). A renewal process in which the interrenewal times have a PH-distribution is called <u>a PH-renewal process</u> [12].

We refer the reader to the cited references for the closure properties of the class of PH-distribution. For use in the sequel, we need the following. The mean of the distribution F(·) is $\mu_1'=-\underline{\alpha}T^{-1}\underline{e}$, and the probability distribution $F^*(x)=(\mu_1')^{-1}\int_0^x[1-F(u)]du$, $x\geq 0$, is also of phase type and its representation is ($\underline{\pi}$,T).

Moreover the stationary version of a PH-renewal process is obtained by starting the Markov process Q^* with the initial probability vector $\underline{\pi}$.

We shall now use the Markov chain Q^* as a substratum for the definition of the point process of interest. A transition from the state i to the state j, which does not involve a renewal (i.e. no visit to the "instantaneous" state m+1) will be called an <u>(i,j)-transition</u>. We recall that in a Markov process, transitions from a state to itself are not considered, so that we only have (i,j)-transitions for $i \neq j$. A transition from i to j, which involves a renewal, will be called an <u>(i,j)-renewal transition</u>. Such transitions which, rigorously defined, go from i to the instantaneous state m+1 and thence to j, may go from a state to itself. Whenever the process is in state i at time t, an (i,j)-renewal transition occurs with the elementary probability $T_i^\circ \alpha_j dt$.

3. The Point Process

For reasons, which will be apparent in the examples of applied interest, we shall allow each of the events in the process to be <u>multiple</u> and we shall use the intuitively appealing terminology of <u>group arrivals</u>. There are three different types of arrival epochs, for which we now define appropriate notation:

a. During any sojourn time of the Markov process Q^* in the state i, $1 \le i \le m$, there are Poisson arrivals of rate λ_i and group size density $\{p_i(k),\ k \ge 0\}$. We shall denote the probability generating function of $\{p_i(k)\}$ by $\phi_i(z)$ and we may assume without loss of generality that $\phi_i(0)=0$, for $1 \le i \le m$. If there are no arrivals of the other types, the doubly

stochastic Poisson process [2] so obtained is called the Markov-modulated Poisson process.

b. At (i,j)-renewal transitions, there are group arrivals with probability density $\{r_{ij}(k), k\geq 0\}$. The generating function of the density $\{r_{ij}(k)\}$ is denoted by $\phi_{ij}(z)$, for $1\leq i,j\leq m$. The matrix $\{\phi_{ij}(z)\}$ is denoted by $\phi(z)$ and we do allow $\phi_{ij}(0)$ to be positive for some (i,j).

c. At (i,j)-transitions, there are group arrivals with probability density $\{q_{ij}(k), k\geq 0\}$. The generating function of the density $\{q_{ij}(k)\}$ is denoted by $\Psi_{ij}(z)$, for $i\neq j$. We shall find it convenient to define $\Psi_{ii}(z)=1$, for $1\leq i\leq m$. The matrix $\{\Psi_{ij}(z)\}$ is denoted by $\Psi(z)$. We do allow some of the $\Psi_{ij}(0)$, $i\neq j$, to be positive.

We now make the following independence assumptions. For every $t>0$, given the path function of the Markov process Q^*, the epochs of the first type of arrivals are conditionally independent, given the successive sojourn times and behave as a homogeneous Poisson process on every sojourn interval. Given the times and types of the arrival epochs up to time t, the group sizes are conditionally independent and have the probability densities, given above.

By $N(t)$ and $J(t)$, $t\geq 0$, we denote respectively the number of arrivals in $(0,t]$ and the state of the Markov process Q^*. It is then easy to see that the process $\{N(t), J(t), t\geq 0\}$ is a Markov process with the state space $\{k\geq 0\}\times\{1,\ldots,m\}$.

The probabilities $P_{ij}(\nu,t)=P\{N(t)=\nu,\ J(t)=j|N(0)=0,\ J(0)=i\}$ $1\leq i,j\leq m$, are of considerable interest. We derive a recursive

system of differential equations for the matrices $P(\nu,t)=\{P_{ij}(\nu,t)\}$, for $\nu\geq 0$.

The Chapman-Kolmogorov equations for the process $\{N(t),\ J(t),\ t\geq 0\}$ may be written as

$$P'_{ij}(\nu,t)=P_{ij}(\nu,t)(T_{jj}-\lambda_j)+\sum_{k=0}^{\nu} P_{ij}(k,t)\lambda_j p_j(\nu-k)+ \tag{4}$$

$$\sum_{\substack{h=1\\h\neq j}}^{m}\sum_{k=0}^{\nu} P_{ih}(k,t)T_{hj}q_{hj}(\nu-k)+\sum_{h=1}^{m}\sum_{k=0}^{\nu} P_{ih}(k,t)T^{\circ}_{h}\alpha_j r_{hj}(\nu-k),$$

for $1\leq i,j\leq m$ and $\nu\geq 0$. In matrix notation, we have

$$P'(\nu,t)=-P(\nu,t)\Delta(\underline{\lambda})+\sum_{k=0}^{\nu} P(k,t)\Delta(\underline{\lambda})\Delta[\underline{p}(\nu-k)] \tag{5}$$

$$+\sum_{k=0}^{\nu} P(k,t)[T\circ q(\nu-k)]+\sum_{k=0}^{\nu} P(k,t)[T^{\circ}A^{\circ}\circ r(\nu-k)],$$

where $\Delta(\underline{\lambda})=\mathrm{diag}(\lambda_1,\ldots,\lambda_m)$, $\Delta[\underline{p}(\tau)]=\mathrm{diag}\{p_1(\tau),\ldots,p_m(\tau)\}$, for $\tau\geq 0$. The matrix $q(\tau)$ has entries $q_{ij}(\tau)$, for $\tau\geq 0$ and $i\neq j$, but $q_{ii}(\tau)=\delta_{\tau 0}$, for $\tau\geq 0$. The matrix $r(\tau)$ has entries $r_{ij}(\tau)$, for $\tau\geq 0$. The symbol $\circ$ denotes the Schur or entry-wise product of two matrices.

The matrix-generating function $\tilde{P}(z,t)=\sum_{\nu=0}^{\infty} P(\nu,t)z^{\nu}$, defined for $|z|\leq 1$, satisfies the differential equation

$$\frac{\partial}{\partial t}\tilde{P}(z,t)=\tilde{P}(z,t)\{-\Delta(\underline{\lambda})+\Delta(\underline{\lambda})\Delta[\underline{\phi}(z)]+T\circ\Psi(z)+T^{\circ}A^{\circ}\circ\Phi(z)\}, \tag{6}$$

with the initial condition $\tilde{P}(z,0)=I$. It follows that

$$\tilde{P}(z,t)=\exp\{R(z)t\}, \tag{7}$$

where $R(z)=\Delta(\underline{\lambda})\Delta[\underline{\phi}(z)]-\Delta(\underline{\lambda})+T\circ\Psi(z)+T^{\circ}A^{\circ}\circ\Phi(z)$.

We see that $\Psi(1)=\Phi(1)=E$, where $E_{ij}=1$, so that $\tilde{P}(1,t)=\exp(Q^*t)$, as is to be expected. The explicit expression (7) for $\tilde{P}(z,t)$ is also the direct generalization of the formula $\tilde{P}(z,t)=\exp[-\lambda t(1-z)]$, for the ordinary Poisson process.

A. The Mean Matrix

By differentiating with respect to z in (7) and setting z=1, we obtain the matrix

$$M(t)=\left[\frac{\partial}{\partial z}\tilde{P}(z,t)\right]_{z=1}=\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{\nu=0}^{n-1}Q^{*\nu}R'(1)Q^{*n-1-\nu}, \tag{8}$$

where

$$R'(1)=\Delta(\underline{\lambda}\circ\underline{\gamma})+T\circ C+T^{\circ}A^{\circ}\circ D, \tag{9}$$

with $\underline{\gamma}=\underline{\phi}'(1-)$, $C=\psi'(1-)$, and $D=\phi'(1-)$.

We shall first compute the vector $\underline{\mu}(t)=M(t)\underline{e}$, which plays an important role in applications. From (8) we obtain

$$\underline{\mu}(t)=\sum_{n=1}^{\infty}\frac{t^n}{n!}Q^{*n-1}R'(1)\underline{e}, \tag{10}$$

and in [12] we proved that

$$\sum_{n=1}^{\infty}\frac{t^n}{n!}Q^{*n-1}=\int_0^t\exp(Q^*u)du=\Pi t+[I-\exp(Q^*t)](\tau^*\Pi-Q^*)^{-1}, \tag{11}$$

where τ^* is any real number such that $\tau^*\geq\max_i(-Q^*_{ii})$, and Π is the m×m stochastic matrix with identical rows given by $\underline{\pi}$.

It follows that

$$\underline{\mu}(t)=\mu^*\underline{e}t+(I-\Pi)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}+$$

$$[\Pi-\exp(Q^*t)](\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}, \tag{12}$$

where $\mu^*=\underline{\pi}R'(1)\underline{e}$.

We see that $\underline{\pi}\underline{\mu}(t)=\mu^*t$, and that the first two terms in (12) give the linear asymptote of $\underline{\mu}(t)$, since the third term tends to zero as $t\to\infty$. It is advisable therefore to compute the third term separately. Denoting it by $\underline{v}(t)$, it may be evaluated by solving the system of differential equations

$$\underline{v}'(t)=Q^*\underline{v}(t), \qquad \underline{v}(0)=(\Pi-I)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}. \tag{13}$$

It should be emphasized that in many applications of the present point process, the approach to steady-state is slow.

The second term in (12) will be denoted by $\underline{\nu}$ and is given by

$$\underline{\nu}=(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}-\mu^*\tau^{*-1}\underline{e}. \tag{14}$$

The linear asymptote of $EN(t)=\underline{\alpha}\underline{\mu}(t)$, corresponding to the case where the PH-renewal process is an <u>ordinary</u> PH-renewal process is clearly given by $\mu^*t-\mu^*\tau^{*-1}+\underline{\alpha}(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}$.

We shall now discuss the computation of the matrix $M(t)$ and its asymptotic behavior. It readily follows from Formula (8), that

$$M'(t)-M(t)Q^*=\exp(Q^*t)R'(1), \tag{15}$$

with $M(0)=0$, so that

$$M(t)=\int_0^t \exp(Q^*u)R'(1)\exp[Q^*(t-u)]du. \tag{16}$$

Adding $\tau^* M(t)\Pi = \tau^* \Delta[M(t)\underline{e}]\Pi$, to both sides of Equation (15) and evaluating the Laplace transform of both sides of the resulting equation, we obtain

$$\tilde{M}(s) = (sI-Q^*)^{-1}R'(1)(sI+\tau^*\Pi-Q^*)^{-1} + \Delta[\tilde{M}(s)\underline{e}]\tau^*\Pi(sI+\tau^*\Pi-Q^*)^{-1}, \quad (17)$$

in which $\tilde{M}(s) = \int_0^\infty e^{-st}M(t)dt$. Since we know from the theory of regenerative processes, that $\lim_{t\to\infty} t^{-1}M(t)$ exists, this limit may be found by evaluating $\lim_{s\to 0+} s\tilde{M}(s)$. We so obtain

$$\lim_{t\to\infty} t^{-1}M(t) = \mu^*\tau^*\Pi(\tau^*\Pi-Q^*)^{-1} = \mu^*\Pi. \quad (18)$$

By considering the differential equation obtained from (15) for the matrix $M(t)-\mu^*\Pi t$, we obtain the constant term in the asymptotic formula for $M(t)$, since

$$\lim_{t\to\infty} [M(t)-\mu^*\Pi t] = \Pi R'(1)(\tau^*\Pi-Q^*)^{-1} - \mu^*\tau^{*-1}\Pi + \Delta(\underline{\nu})\Pi$$
$$= \Pi R'(1)(\tau^*\Pi-Q^*)^{-1} + (\tau^*\Pi-Q^*)^{-1}R'(1)\Pi - 2\mu^*\tau^{*-1}\Pi. \quad (19)$$

It follows that

$$M(t) = \Pi R'(1)(\tau^*\Pi-Q^*)^{-1} + (\tau^*\Pi-Q^*)^{-1}R'(1)\Pi - 2\mu^*\tau^{*-1}\Pi + \mu^*\Pi t + o(1), \quad (20)$$

as $t\to\infty$.

In computing $M(t)$ numerically, it is again advisable to modify Equation (15) to the equation for the difference $K(t)$ between $M(t)$ and its linear asymptote. We so obtain

$$K'(t) - K(t)Q^* = \exp(Q^*t)R'(1) + \Pi R'(1)(\tau^*\Pi-Q^*)^{-1}Q^* - \mu^*\Pi$$
$$= [\exp(Q^*t)-\Pi]R'(1), \quad (21)$$

where K(0) equals minus the constant term in Formula (20).

B. The Second Moment Matrix

By differentiating twice with respect to z in Formula (7) and setting z=1, we obtain the second factorial moment matrix $M_2(t)$, which is given by

$$M_2(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{\nu=0}^{n-1} Q^{*\nu} R''(1) Q^{*n-1-\nu} +$$

$$2 \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{\nu=0}^{n-2} \sum_{r=0}^{\nu} Q^{*r} R'(1) Q^{*\nu-r} R'(1) Q^{*n-2-\nu}. \tag{22}$$

The matrix $M_2(t)$ can be discussed in an entirely analogous manner as the matrix M(t), but the resulting expressions are much more involved. We shall only discuss the vector $\underline{\mu}_2(t)=M_2(t)\underline{e}$, which is needed in the computation of the variance of the number of arrivals in (0,t).

We have

$$\underline{\mu}_2(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} Q^{*n-1} R''(1)\underline{e} + 2 \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{r=0}^{n-2} Q^{*r} R'(1) Q^{*n-2-r} R'(1)\underline{e}. \tag{23}$$

The first term is given by

$$\mu_2^* t\underline{e} + [I - \exp(Q^*t)](\tau^*\Pi - Q^*)^{-1} R''(1)\underline{e}, \tag{24}$$

where $\mu_2^* = \underline{\pi} R''(1)\underline{e}$, by application of Formula (11).

In order to evaluate the second term, we notice that

$$\sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{r=0}^{n-2} Q^{*r} R'(1) Q^{*n-2-\nu} (\tau^*\Pi - Q^*)$$

$$= \tau^* \sum_{n=2}^{\infty} \frac{t^n}{n!} Q^{*n-2} R'(1)\Pi - \sum_{n=2}^{\infty} \frac{t^n}{n!} \sum_{r=0}^{n-2} Q^{*r} R'(1) Q^{*n-1-r}$$

$$=-M(t)+\sum_{n=1}^{\infty}\frac{t^n}{n!}Q^{*n-1}R'(1)+\tau^*\sum_{n=2}^{\infty}\frac{t^n}{n!}Q^{*n-2}R'(1)\Pi.$$

By repeated application of Formula (11), we obtain that

$$\sum_{n=2}^{\infty}\frac{t^n}{n!}\sum_{r=0}^{n-2}Q^{*r}R'(1)Q^{*n-2-\nu}=-M(t)(\tau^*\Pi-Q^*)^{-1}+\mu^*\frac{t^2}{2}\Pi$$

$$+t(I-\Pi)(\tau^*\Pi-Q^*)^{-1}R'(1)\Pi+t\Pi R'(1)(\tau^*\Pi-Q^*)^{-1}$$

$$+[I-\exp(Q^*t)]\{(\tau^*\Pi-Q^*)^{-1}R'(1)(\tau^*\Pi-Q^*)^{-1}-(\tau^*\Pi-Q^*)^{-2}R'(1)\Pi\}.$$

Upon substitution in Formula (23), we obtain

$$\underline{\mu}_2(t)=-2M(t)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}+\mu^{*2}t^2\underline{e}+\mu_2^*t\underline{e} \tag{25}$$

$$+2t\mu^*(I-\Pi)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}+2t\pi R'(1)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}$$

$$+[I-\exp(Q^*t)]\{(\tau^*\Pi-Q^*)^{-1}R''(1)\underline{e}-2\mu^*(\tau^*\Pi-Q^*)^{-2}R'(1)\underline{e}$$

$$+2(\tau^*\Pi-Q^*)^{-1}R'(1)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}\}$$

By using Formula (20), we obtain the following asymptotic expansion for $\underline{\mu}_2(t)$ as $t\to\infty$.

$$\underline{\mu}_2(t)=t^2\mu^{*2}\underline{e}+t\{\mu_2^*\underline{e}+2\mu^*(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}-4\mu^{*2}\tau^{*-1}\underline{e}$$

$$+2[\underline{\pi}R'(1)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}]\underline{e}\} \tag{26}$$

$$+4\mu^{*2}\tau^{*-2}\underline{e}-\mu_2^*\tau^{*-1}\underline{e}+(\tau^*\Pi-Q^*)^{-1}R''(1)\underline{e}$$

$$-2[\underline{\pi}R'(1)(\tau^*\Pi-Q^*)^{-2}R'(1)\underline{e}]\underline{e}-2\mu^*\tau^{*-1}(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}$$

$$+2[(\tau^*\Pi-Q)^{-1}R'(1)]^2\underline{e}-2\tau^{*-1}[\underline{\pi}R'(1)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}]\underline{e}$$

$$-2\mu^*(\tau^*\Pi-Q^*)^{-2}R'(1)\underline{e}-2\mu^*\tau^{*-1}(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}$$

$$+o(1).$$

We also have that

$$\underline{\pi}\underline{\mu}_2(t)=\mu^{*2}t^2+\mu_2^*t-2\mu^{*2}\tau^{*-1}t+2t\underline{\pi}R'(1)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}$$
$$-2\underline{\pi}R'(1)[I-\exp(Q^*t)](\tau^*\Pi-Q^*)^{-2}R'(1)\underline{e}. \tag{27}$$

Remarks

Although the preceding formulas are complicated, they are well-suited for numerical computation, as they involve a small number of vectors which need to be evaluated only once. In the few examples where these coefficient vectors may be explicitly evaluated, they expand into very complicated algebraic functions of the moments of $F(\cdot)$ and the group size densities. We shall illustrate this below by an example, which is of particular interest.

If one wishes to compute the vectors $\underline{\mu}_1(t)$ and $\underline{\mu}_2(t)$ over an interval $[0,T]$, it is advisable to implement a numerical method for the solution of systems of differential equations to evaluate the three items $\underline{\mu}_1(t)$, $M(t)$ and $\underline{\mu}_2(t)$ for $t=n\Delta$, for successive values of n and for a step Δ. In this manner $\exp(Q^*t)$ needs to be evaluated only once and can be substituted in all the formulas where it occurs.

It is also worth stressing that one can frequently avoid computing the matrix $M(t)$. In order e.g. to compute the variance time curve corresponding to the ordinary PH-renewal process, we need to evaluate $\underline{\alpha}\underline{\mu}(t)$ and $\underline{\alpha}\underline{\mu}_2(t)$. It is easy to see that the formula for $\underline{\alpha}\underline{\mu}_2(t)$ only involves the vector $\underline{z}(t)=\underline{\alpha}M(t)$, which can be computed by solving the differential equations

$$\underline{z}'(t)-\underline{z}(t)Q^*=\underline{\alpha}\exp(Q^*t)R'(1),$$

with $\underline{z}(0)=0$. This involves far fewer arithmetic operations and less storage then the computation of $M(t)$.

From the mean and the second factorial moments, it is of course a routine matter to compute the variances.

4. Examples

A. The Superposition of a Poisson Process and a PH-renewal Process

Kuczura [5] considers a queue whose input process is the superposition of a Poisson and a renewal process, but in order to obtain tractable transform solutions he imposed the requirement that the distribution of the interrenewal times have a rational Laplace-Stieltjes transform. This is only slightly more general than requiring the renewal process to be of phase type.

We believe that there is practical merit to consider queues, whose arrival process is the superposition of a Poisson process (background input) and a PH-renewal process with group arrivals (burst inputs). Such a process corresponds to a given matrix T and vector $\underline{\alpha}$ and the parameter choices $\lambda_i=\lambda$, $\phi_i(z)=z$, for $1\leq i\leq m$, $\Phi(z)=p(z)E$, $\Psi(z)=E$, where $p(z)$ is the probability generating function of the group sizes in the renewal arrival process.

For this case, we shall evaluate the coefficient vectors which appear in the formulas for $\underline{\mu}_1(t)$ and $\underline{\mu}_2(t)$. We do not recommend that these explicit formulas be used in numerical computations, but we use them to illustrate the complicated dependence of these coefficient vectors on the moments of $F(\cdot)$ and the group size density. By particularizing further to the case of an ordinary PH-renewal process, we shall also be able to check our computations against known formulas for the asymptotic

variance of a renewal process.

We obtain the particular formulas:

$$R'(1)=\lambda I+\eta_1 T^\circ A^\circ, \qquad \eta_1=p'(1), \tag{28}$$

$$R''(1)=\eta_2 T^\circ A^\circ, \qquad \eta_2=p''(1),$$

$$R'(1)\underline{e}=\lambda\underline{e}+\eta_1\underline{T}^\circ, \qquad R''(1)\underline{e}=\eta_2\underline{T}^\circ$$

$$\underline{\pi}R'(1)=\lambda\underline{\pi}+\eta_1\mu_1'^{-1}\underline{\alpha}, \qquad \underline{\pi}R''(1)=\eta_2\mu_1'^{-1}\underline{\alpha}$$

$$\mu_1^*=\lambda+\eta_1\mu_1'^{-1}, \qquad \mu_2^*=\eta_2\mu_1'^{-1}$$

The proof of the following three formulas requires some interesting algebraic manipulations for PH-distributions. The first of these formulas is proved below and the other two follow by similar calculations.

$$(\tau^*\Pi-Q^*)^{-1}\underline{T}^\circ=\mu_1'^{-1}T^{-1}\underline{e}+\mu_1'^{-1}(\tau^{*-1}+\tfrac{1}{2}\mu_2'\mu_1'^{-1})\underline{e}, \tag{29a}$$

$$\underline{\alpha}(\tau^*\Pi-Q^*)^{-1}=\underline{\pi}T^{-1}-\mu_1'^{-1}(\tau^{*-1}+\tfrac{1}{2}\mu_2'\mu_1'^{-1})\underline{\alpha}T^{-1}, \tag{29b}$$

$$(\tau^*\Pi-Q^*)^{-1}T^{-1}\underline{e}=-T^{-2}\underline{e}-\tfrac{1}{2}\mu_2'\mu_1'^{-1}T^{-1}\underline{e}$$

$$-(\tfrac{1}{2}\mu_2'\mu_1'^{-1}\tau^{*-1}+\tfrac{1}{4}\mu_2'^2\mu_1'^{-2}\tau^{*-1}-\tfrac{1}{6}\mu_3'\mu_1'^{-1})\underline{e}. \tag{29c}$$

In order to prove (29a), denote the left hand side by $\underline{u}$ and write

$$(\tau^*\Pi-T-T^\circ A^\circ)\underline{u}=\underline{T}^\circ,$$

or

$$\tau^*(\underline{\pi}\underline{u})\underline{e}-T\underline{u}-(\underline{\alpha}\underline{u})\underline{T}^\circ=\underline{T}^\circ,$$

which leads to

$$\underline{u}=\tau^*(\underline{\pi}\underline{u})T^{-1}\underline{e}+(1+\underline{\alpha}\underline{u})\underline{e}, \tag{30}$$

since $T^{-1}\underline{T}^\circ=-\underline{e}$. Recalling that $\mu_1'=-\underline{\alpha}T^{-1}\underline{e}$, premultiplication by $\underline{\alpha}$ in the preceding equation leads to $\underline{\pi}\underline{u}=\mu_1'^{-1}\tau^{*-1}$.

Noting that $-\underline{\pi}T^{-1}\underline{e}$ is the mean of $F^*(x)=\mu_1'^{-1}\int_0^x[1-F(u)]du$ we readily obtain that $\underline{\pi}T^{-1}\underline{e}=-\frac{1}{2}\mu_2'\mu_1'^{-1}$. Premultiplication by $\underline{\pi}$ in (30) then yields that

$$1+\underline{\alpha}\,\underline{u}=\mu_1'^{-1}(\tau^{*-1}+\tfrac{1}{2}\mu_2'\mu_1'^{-1}),$$

and Formula (29a) follows upon substitution.

Routine, but tedious calculations now yield the following explicit expressions for coefficient vectors appearing in Formula (25).

$$(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}=\eta_1\mu_1'^{-1}T^{-1}\underline{e}+[\lambda\tau^{*-1}+\eta_1\mu_1'^{-1}(\tau^{*-1}+\tfrac{1}{2}\mu_2'\mu_1'^{-1})]\underline{e}, \qquad (31a)$$

$$\underline{\mu}(t)=(\lambda+\eta_1\mu_1'^{-1})t+\eta_1\mu_1'^{-1}[I-\exp(Q^*t)]T^{-1}\underline{e}, \qquad (31b)$$

$$M(t)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}=\eta_1\mu_1'^{-1}M(t)T^{-1}\underline{e}\ +$$
$$[\lambda\tau^{*-1}+\eta_1\mu_1'^{-1}(\tau^{*-1}+\tfrac{1}{2}\mu_2'\mu_1'^{-1})]\underline{\mu}(t), \qquad (31c)$$

$$(I-\Pi)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}=\eta_1\mu_1'^{-1}T^{-1}\underline{e}+\tfrac{1}{2}\eta_1\mu_2'\mu_1'^{-2}\underline{e}, \qquad (31d)$$

$$\Pi R'(1)(\tau^*\Pi-Q^*)^{-1}R'(1)\underline{e}=$$
$$\{(\lambda+\eta_1\mu_1'^{-1})(\lambda\tau^{*-1}+\eta_1\mu_1'^{-1}\tau^{*-1}+\tfrac{1}{2}\eta_1\mu_2'\mu_1'^{-2})\ -$$
$$\eta_1^2\mu_1'^{-1}-\tfrac{1}{2}\lambda\eta_1\mu_2'\mu_1'^{-2}\}\underline{e}, \qquad (31e)$$

$$(\tau^*\Pi-Q^*)^{-1}R''(1)\underline{e}=\eta_2\mu_1'^{-1}T^{-1}\underline{e}+\eta_2\mu_1'^{-1}(\tau^{*-1}+\tfrac{1}{2}\mu_2'\mu_1'^{-1})\underline{e}, \qquad (31f)$$

$$(\tau^*\Pi-Q^*)^{-2}R'(1)\underline{e}=[\lambda\tau^{*-2}+\eta_1\mu_1'^{-1}\tau^{*-1}(\tau^{*-1}+\tfrac{1}{2}\mu_2'\mu_1'^{-1})]\underline{e}$$
$$+\eta_1\mu_1'^{-1}[-T^{-2}\underline{e}-\tfrac{1}{2}\mu_2'\mu_1'^{-1}T^{-1}\underline{e}\ -$$
$$(\tfrac{1}{2}\mu_2'\mu_1'^{-1}\tau^{*-1}+\tfrac{1}{4}\mu_2'^2\mu_1'^{-2}-\tfrac{1}{6}\mu_3'\mu_1'^{-1})\underline{e}], \qquad (31g)$$

$$[(\tau^*\Pi-Q^*)^{-1}R'(1)]^2\underline{e}=(\lambda\tau^{*-1}+\eta_1\mu_1'^{-1}\tau^{*-1}+\tfrac{1}{2}\eta_1\mu_2'\mu_1'^{-2})^2\underline{e}$$

$$+2\eta_1\mu_1'^{-1}[-T^{-2}\underline{e}-\tfrac{1}{2}\mu_2'\mu_1'^{-1}T^{-1}\underline{e}$$

$$-(\tfrac{1}{2}\mu_2'\mu_1'^{-1}\tau^{*-1}+\tfrac{1}{4}\mu_2'^2\mu_1'^{-2}-\tfrac{1}{6}\mu_3'\mu_1'^{-1})\underline{e}]. \tag{31h}$$

Substitution into Formula (25) does not produce any substantial cancellation of terms, other than that resulting from $[I-\exp(Q^*t)]\underline{e}=\underline{0}$. In order to check on the accuracy of our derivations however, we substituted the expressions (31) into the formula for $\underline{\pi\mu}_2(t)$ and obtained

$$\underline{\pi\mu}_2(t)=(\lambda+\eta_1\mu_1'^{-1})^2t^2+(\eta_2\mu_1'^{-1}+\eta_1^2\mu_2'\mu_1'^{-3}-2\eta_1^2\mu_1'^{-1})t$$

$$+\tfrac{1}{2}\eta_1^2\mu_2'^2\mu_1'^{-4}-\tfrac{1}{3}\eta_1^2\mu_3'\mu_1'^{-3}. \tag{32}$$

If one sets η_1 1, $\eta_2=0$, and evaluates <u>the variance</u> $\mathrm{Var}_e[N(t)]$ of $N(t)$ for the stationary process, one obtains in terms of the central moments of $F(\cdot)$ that

$$\mathrm{Var}_e[N(t)]=(\lambda+\sigma^2\mu_1'^{-3})t+\tfrac{1}{6}+\tfrac{1}{2}\sigma^4\mu_1'^{-4}-\tfrac{1}{3}\mu_3\mu_1'^{-3}, \tag{33}$$

which is in agreement with Formula (18), p. 58 of Cox [1].

B. The Markov-Modulated Poisson Process

If $\Phi(z)=\Psi(z)=E$, and $\phi_i(z)=z$, for $1\le i\le m$, we obtain an interesting example of a doubly stochastic Poisson process, which was used as an arrival process in the queueing models, discussed in [6], [13], [14] and [15]. Since in this case $R'(1)=\Delta(\underline{\lambda})$, and $R''(1)=0$, some minor simplifications occur in the general moment formulas.

The Markov-modulated Poisson process can be used to model a large variety of qualitative phenomena, such as interruptions in arrivals, rush-hour behavior and others. Heffes [3], in a telephone engineering model, considered the interrupted Poisson process in which arrivals occur on alternating, exponentially distributed intervals. This corresponds to the particular choice

$$T = \begin{vmatrix} -\sigma_1 & \sigma_1 \\ 0 & -\sigma_2 \end{vmatrix}, \qquad \underline{T}^\circ = \begin{vmatrix} 0 \\ \sigma_2 \end{vmatrix}, \qquad \underline{\alpha}=(1,0),$$

$$\lambda_1=\lambda,\ \phi_1(z)=z,\ \lambda_2=0,\ \phi_2(z) \text{ arbitrary},\ \Phi(z)=\Psi(z)= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}.$$

A more general, useful situation arises if we define an interrupted Poisson process on an alternating renewal process of phase type. Let the odd-and-even-numbered intervals have PH-distributions $F_1(\cdot)$ and $F_2(\cdot)$ with irreducible representations $(\underline{\alpha}_1, T_1)$ and $(\underline{\alpha}_2, T_2)$ of dimensions m_1 and m_2 respectively, then a natural extension of the preceding example is given by

$$T = \begin{vmatrix} T_1 & T_1^\circ A_2^\circ \\ 0 & T_2 \end{vmatrix}, \qquad \underline{T}^\circ = \begin{vmatrix} \underline{0} \\ \underline{T}_2^\circ \end{vmatrix}, \qquad \underline{\alpha}=(\underline{\alpha}_1,\underline{0}),$$

$$\lambda_i=\lambda, \qquad \lambda_i=0,$$

$$\phi_i(z)=z, \text{ for } 1\le i\le m_1, \qquad \phi_i(z) \text{ arbitrary for } m_1+1\le i\le m_1+m_2,$$

$$\Phi(z)=\Psi(z)=E, \qquad \text{with } E_{ij}=1.$$

In the expression for T, the square matrices T_1 and T_2 are of orders m_1 and m_2 respectively, $A_2^\circ=\Delta(\underline{\alpha}_2)$ and T_1° is an $m_1 \times m_2$ matrix, with all its columns equal to $\underline{T}_1^\circ$. It is easily verified

that the matrix Q*, which is given

$$Q^* = \begin{vmatrix} T_1 & T_1^\circ A_2^\circ \\ T_2^\circ A_1^\circ & T_2 \end{vmatrix},$$

has the stationary probability vector

$$\underline{\pi} = [\mu_1'(\mu_1'+\mu_2')^{-1}\underline{\pi}_1,\ \mu_2'(\mu_1'+\mu_2')^{-1}\underline{\pi}_2],$$

where μ_k' is the mean of $F_k(\cdot)$, k=1,2, and $\underline{\pi}_k$ is given by

$\underline{\pi}_k(T_k+T_k^\circ A_k^\circ)=\underline{0}$, $\underline{\pi}_k\underline{e}=1$, for k=1,2.

Alternating periods of rush and slack arrivals can of course be modelled by other choices of the λ_i-parameters.

C. The Markov Arrival Process

If $\underline{\lambda}=\underline{0}$, $\phi(z)=zE$, and $\Psi_{ij}(z)=z$, for $i\neq j$, we obtain a particular semi-Markovian arrival process in which the underlying Markov renewal process is a Markov process.

By suitably enlarging the state space, any semi-Markov arrival process in which the sojourn time distributions are of phase type can in fact be obtained as a particular case of our process by allowing arrivals at some (i,j)-transitions and not at others. This is however rarely of practical interest in view of the high order of the Q*-matrix, usually required by this construction.

D. Arrivals Inhibited or Stimulated by Renewals

Prof. E. Gelenbe renewed our interest in the point processes, discussed here, by describing practical queueing situations in which e.g. substantial group arrivals at renewal epochs inhibit

background arrivals for a certain length of time after their arrival. Although there are a variety of ways of modelling input sequences of this type, it is not easy to do so in an analytically or computationally tractable way. We can easily choose particular cases of the present point process which can serve as tractable qualitative models for such arrival streams.

A PH-distribution will be called progressive, if it has a representation ($\underline{\alpha}$,T) in which the matrix T is upper triangular. It is easy to see that a PH-distribution is progressive if and only if it is a finite mixture of generalized Erlang distributions. Every path function of the Markov process Q is then non-decreasing. We can allow e.g. group arrivals at some or all of the (i,j)-renewal transitions and select the λ_i-parameters of the states close to renewals so as to model the inhibitory effect of the group arrivals.

Although the statistical problems, related to fitting such models to observational data, require much further research, the ease with which the mean and variance time curves can be computed for the point processes discussed here should be useful in fitting sample mean and variance curves. Computer graphical methods appear to be very promising for such purpose and are currently being investigated.

5. The Covariance Structure

In this section we consider the covariance of the random variables $N(t)$ and $N(t+t_1+t')-N(t+t_1)$, where $t>0$, $t_1\geq 0$, $t'>0$, for the stationary version of the underlying PH-renewal process. For the non-stationary version similar, but more complicated formulas,

are routinely obtained.

We readily see that

$$E\{z_1^{N(t)} z_2^{N(t+t_1+t')-N(t+t_1)} I[J(t+t_1+t')=j]|J(0)=i\},$$

is given by the (i,j)-entry of the matrix

$$\exp[R(z_1)t]\exp(Q^*t_1)\exp[R(z_2)t'].$$

Differentiating with respect to z_1 and with respect to z_2, setting $z_1=z_2=1$, we readily see that the covariance of N(t) and $N(t+t_1+t')-N(t+t_1)$ is given in the stationary case by

$$\mathrm{Cov}=\underline{\pi}M(t)\exp(Q^*t_1)M(t')\underline{e}-\mu^*tt'. \qquad (34)$$

Since $M(t')\underline{e}=\underline{\mu}(t')$ is given by Formula (12) and

$$\underline{\pi}M(t)=\mu^*t\underline{\pi}-\mu^*\tau^{*-1}\underline{\pi}+\underline{\pi}R'(1)(\tau^*\Pi-Q^*)^{-1}$$

$$+\underline{\pi}R'(1)[\Pi-\exp(Q^*t)](\tau^*\Pi-Q^*)^{-1}, \qquad (35)$$

and using the fact that $(\tau^*\Pi-Q^*)^{-1}$ and $\exp(Q^*u)$ commute, we obtain after routine calculations that

$$\mathrm{Cov}=\underline{\pi}R'(1)[I-\exp(Q^*t)]\exp(Q^*t_1)[I-\exp(Q^*t')](\tau^*\Pi-Q^*)^{-2}R'(1)\underline{e}. \qquad (36)$$

For the particular case of Example A, we obtain by applying Formula (31g) that

$$\mathrm{Cov}=-n_1^2\mu_1'^{-2}\underline{\alpha}[I-\exp(Q^*t)]\exp(Q^*t_1)[I-\exp(Q^*t')](T^{-2}\underline{e}+\tfrac{1}{2}\mu_2'\mu_1'^{-1}T^{-1}\underline{e}). \quad (37)$$

We see that the numerical computation of the covariance is again reduced to the solution of simple systems of differential equations.

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